Some Suggestions for the Exam

Math 116-017, Rafe Kinsey, February 3, 2012

Here's a *non-exhaustive* list of some tips that came to mind about the material.
I can have extra office hours before the exam: email me, and we can figure out a time.

Chapter 8 Technique

Here are some guidelines for how you should go about solving the sorts of problems in chapter 8. I **strongly recommend** that you follow these steps; those who do so are much more likely to get correct answers.

Follow Instructions! If you are told in a subpart to find the work for a given slice, then do this—don't do the Riemann sum or integral until the next part of the problem.

1. Figure out what you are calculating. There are lots of cases where that thing is easy to do **if** certain things are constant. E.g.

   - Work = force times distance if distance is constant
   - mass is density times volume if the density is constant
   - the force on something[1] is pressure times area, if pressure is constant
   - Distance traveled is velocity times time, if velocity is concept.

   Note that the process below will be variations of the same idea, no matter which of these examples (or others) comes up.

2. Chop things up into "slices" (sometimes these will be vertical or horizontal slices, sometimes they may be annular shells or other things): **we choose slices where the quantities above are approximately constant on these thin slices**, this way we can use the easy formulas above to figure out the quantity desired on that slice. For example:

   - For the work required to lift a liquid to the top of a container, different parts of the liquid are going to have to be lifted different distances. So if we choose horizontal slices of the liquid, we'll have approximately the same distance lifted for each slice.
   - If we're told that the density depends on radial distance, we'll want to choose shells.

3. Choose a **good variable**, **draw your slice**, and **draw exactly what your variable is: label various points**. **I'M GOING TO USE OBNOXIOUS CAPITAL LETTERS TO EMPHASIZE THIS: IT'S A VERY GOOD IDEA TO DO THIS, TO LABEL WELL, and IN PARTICULAR TO LABEL PRECISELY WHAT YOUR VARIABLE IS, WITH SEVERAL POINTS.** E.g., if you have that the variable h goes from the ground when $h = 0$ to the top of a building when $h = 100$, then draw that explicitly. **Remember**: if you are told in the problem which variable to use (e.g., if you are told the density is a function of y, where y is distance north or south of a highway), then **you have to** use that variable if you're asked to give answers in terms of that variable. If not, often there are multiple choices of variable that can work fine—as long as you're consistent.

[1] Don't get confused between our two types of problems with force. In work problems, the force corresponds to the *weight* of the object, it's the force required to lift it. In pressure problems, we use pressure times area to calculate the force *on* an object.

4. If you want to calculate some quantity on the slice, you have to calculate the subsidiary quantities in terms of your variable. E.g.

   - To calculate work, you'll need to find the weight of your slice times the distance lifted of that slice, as a function of your variable.
   - To calculate the mass of an object, you'll need a density δ for that slice, times the volume or area or length of that slice (depending on what the units of density are)

   This is a key and difficult step. It's best done when done systematically:

   - **Draw a careful picture of your slice**, label carefully precisely what the relevant quantities and variables are.
   - Often you will be using basic geometry for these things. Sometimes this will be similar triangles, sometimes this will be using the Pythagorean theorem, or other parts of trigonometry[2] or other results about circles or spheres. Be careful to distinguish between radius and diamter, and other issues like this.
   - Use units to check your work (especially for calculating densities or centers of mass).
   - If these subsidiary quantities are linear functions (this is true for things like calculating widths of triangles or trapezoids[3] but not things like circles), you can calculate this function by finding two points and using the point-slope formula.
   - **IMPORTANT**: Double-check that what you've gotten for these formulas is right on a few points: check the top bottom, the bottom point, and a middle point. Do these things work out right?

5. Now multiply these subsidiary quantities together to get the answer for a slice. Once again **pay attention to units** as a way of double-checking that you've done the right thing (and don't forget the Δx or Δh or whatever has units)

6. Turn this sum into a Riemann sum.[4]

7. Turn this sum into an integral. The limits of integration should be the values in your variable of where the slices range from: if your slices range from $x = 3$ to $x = 5$, then you should end up with an integral $\int_3^5 f(x)dx$. (This is important. For example, if you're lifting the water that's in the bottom 5 feet of a container up to a height of 100 feet, then your integral will be $\int_0^5 \cdots dh$, where h is height from the bottom; the 100 will appear in the distance term.[5]

8. Double-check that things make sense in the end. It's easy to accidentally flip things around so that you get a negative answer when something should be positive, or something like that. Also: there are some problems where you have to split the problem into two parts: maybe the function will look different when $y < 0$ from when $y > 0$ (as in quiz 4, where we needed the $|y|$).

Note: There are different variations of these problems—we've seen some of the subtleties in various quiz and team homework problems. The key is to understand this process above: if you follow this process, you should be able to figure out problems that are different from what you've seen before.

[2]You should remember the unit circle, 30-60-90 triangles, etc.

[3]Note that doing an analogous "similar trapezoid" approach doesn't work so well (it can work, but only if the trapezoids are similar, which isn't as often).

[4]Note: you just need to write $\sum f(x)\Delta x$ for the Riemann sum if asked, you don't need to write more (there aren't limits of integration for the Riemann sum).

[5]If you made a different choice of variable, you might end up with an integral $\int_{95}^{100} \ldots dk$, for example, which would be if you chose a variable scale where the water started between $k = 95$ and $k = 100$.

Integration Techniques

Definitely do some practice for this. If you've struggled with the gateway, do some practice gateway tests (even if you passed already).

An important trick that occasionally comes up:

$$\int \cos x \sin x dx = -\cos^2 x - \int \sin x \cos x dx$$

Here we made the integration by parts $u = \cos x, du = \sin x dx$. Now we bring the $-\int sinx \cos x dx$ over the other side (and put in a $+C$) to get

$$2\int \cos x \sin x dx = -\cos^2 x + C$$

Therefore

$$\int \cos x \sin x dx = -1/2 \cos^2 x + C$$

As you've seen in your practice: sometimes integrating by parts multiple times, or doing a combination of integration by parts and substitution, works.

But remember: Sometimes you can solve an integral much more easily, just using the rules for polynomials or other simple functions. Always check to see if you can do something simple like this first!

Also: it's a good idea to have the basic antiderivatives on your index card.

Some odds and ends

- Review the units for work on page 424—it's a good idea to put these down on your index card. Remember also: you need to multiply by g to turn SI mass units (e.g. g) into weight units.
- Review old quiz problems, team homework problems, and exam problems: these will give you models for what exam problems are like. **Just looking at solutions won't help**: you have to try to do the problems by yourself, without looking at the solution. Afterwards look at the solution. If you got it wrong, try to understand the solution, then put it away, and try to do the problem again without looking at the solution.
- The "Check your understanding" problems at the end of chapters are good practice for the sorts of true/false questions that often pop up on exams.
- **Also: make sure you understand what the fundamental theorems of calculus mean.** Remember, work slowly, write things down precisely, turn it into things we know about.
- It's a good idea to do some practice "draw an anti-derivative of this function" problems. It's very easy to make these problems for yourself. Make sure to draw a function that satisfies all the information you know (both in terms of maxima/minima, concavity; in terms of when you have sharp corners; in terms of the amount of displacement if you know the area, using the first fundamental theorem)
- Remember the techniques we've used for averages—sometimes our intuition fails us, we have to use the formulas, which we know are right. **Similarly**: for ch-8 style problems, our intuition can fail us, it's important to draw slices, calculate things on slices first.
- When we're asked to estimate some quantity where we only have partial information, we'll have to use LHS or RHS or trapezoid rule or midpoint rule. Make sure to do it for the appropriate integrand. For example, if you know the function $f(x)$ for some values of x, and you need to estimate $\int_2^7 f(x)^3 dx$, then you need to use one of these approximations for the function $f(x)^3$, not the function $f(x)$. (We can calculate $f(x)^3$ for each x that we know $f(x)$ of.)

- Remember the rules for when LHS and RHS are under or overestimates (increasing/decreasing) and for midpoint and trapezoid rules (concavity).
- When doing density, remember to think about what the units are: $kg/m, kg/m^2, and kg/m^3$ all mean that you're looking for different things to multiply it by (length, area, volume)
- For center-of-mass problems: remember to calculate all the coordinates $\overline{x}, \overline{y}, \overline{z}$ (and label them). Often several of these will be easy to do by symmetry. The $A_x(x)dx$ should together have the right units for your density (and remember the dx has units).
- Note that the solutions for old webHW problems are online. It's a good idea to review them if you had trouble on a problem.
- Sometimes you have problems with constants in them. Remember: our rules for integration work fine with constants. Just solve the problem in terms of the constants.
- To find the intersection of two graphs, you should do this by hand, rather than using your calculator. E.g., if you want to find when $f = x^2$ and $g = 5x + 1$ intersect, set $x^2 = 5x + 1$, and then solve using the quadratic formula.

Good luck! Make sure to get a good night's sleep and take care of yourself before the exam.