# Series Convergence Problems (9.4) Worked Out

Rafe Kinsey

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Note: for some of these examples, there are other tests you can use as well. Sometimes different tests work equally well. Other times, two different tests work, but one will be much easier than another. In that case, it's a good idea to figure out which one will be easier, so you'll spend less time.

A good studying strategy is to think about all the different tests, why one might or might not work, which of the ones that work will be easy, which will be hard, etc.

If you didn't do these problems, **make sure to do them on your own**, don't just read the answers and think you can do them. If you struggle, look at the answers, think about it, put it away, come back 30 minutes later and try it without looking at the answers.

I haven't gone through all the details of canceling out factorials. If you're struggling with doing this, come talk to me.

**Warning**: Be careful about saying things are "like" such and such without justifying. Make sure you use a test to justify fully your arguments! Sometimes your intuitions are wrong, that's why you have to test it.

## 56

$\sum_{n=0}^{\infty} \frac{(0.1)^n}{n!}$. A good test is the ratio test. Here $a_n = \frac{(.1)^n}{n!}$. Thus

$$\frac{a_{n+1}}{a_n} = \frac{(.1)^{n+1}/(n+1)!}{(.1)^n/n!}.$$

Therefore,

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(.1)^{n+1}/(n+1)!}{(.1)^n/n!} = \lim_{n\to\infty} \frac{.1}{n+1} = 0.$$

Thus, the ratio test ensures that this converges.

## 60

$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

Let $a_n = \frac{(2n)!}{(n!)^2}$. We have

$$\frac{\frac{2(n+1))!}{(n+1)!}^2}{\frac{(2n)!}{(n!)^2}} = \frac{(2n+2)!}{(2n)!}\frac{(n!)^2}{((n+1)!)^2}$$

$$= \frac{(2n+1)(2n+2)}{(n+1)^2}$$

Therefore

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(n+1)^2} = \lim_{n\to\infty} \frac{4n^2+3n+2}{n^2+2n+1} = 4.$$

By the ratio test, therefore, this diverges.

**Note**: Here's a good example of where being sloppy with parentheses can lead you astray. The first time I did this, I was careless and I wrote $2n+1$, rather than $2n+2$, giving me the wrong answer. **Be careful with parentheses!**

**64**

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{3n-1}}$$

For this we use the alternating series test. It's clearly alternating, and $0 < \frac{1}{\sqrt{3(n+1)-1}} < \frac{1}{\sqrt{3-1}}$ for all $n$. Thus it converges. (Note, however, that it converges only *conditionally*; we could see this by using, for example, the integral test, with $\int_1^\infty \frac{1}{\sqrt{3x-1}}dx$, which diverges.)

**66**

$\sum_{n=1}^\infty \frac{\sin n}{n^2}$

Here the best thing to do is the comparison test. Let's compare this with $\sum_{n=1}^\infty \frac{1}{n^2}$. There is one problem, however: $\sin n$ isn't always positive, and the comparison test requires all the terms to be positive. So first let's show that $\sum |\sin n|\, n^2$ converges, and then by Theorem 9.6 we'll know that the original series converges also (absolute convergence implies convergence). It's clear that $|\sin n|\,/n^2 \leqslant \frac{1}{n^2}$. And we know that $\sum_{n=1}^\infty \frac{1}{n^2}$ converges. (Note that we did the correct order for the comparison test.) Therefore $\sum_{n=1}^\infty |\sin n|\,/n^2$ converges. And so we've got what we want.