Comparison Problems Worked Out

Rafe Kinsey

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NOTE: A lot of you struggled with this (and some of you who didn't do this exercise might also have difficulties). It's important to follow the instructions of the comparison test.

Step 1: make a guess about convergence or divergence. Step 2, **follow the instructions** of the comparison tests to find an *appropriate* comparison function. Sometimes the obvious choice (something like $\frac{1}{x^p}$) will work. Sometimes (as was the case in problems 11 and 18), that obvious choice doesn't satisfy the inequality we want. **Make sure to read carefully the comparison test, understand why you need different inequalities for convergence and divergence.** What you **have to do** is find a new comparison function, usually by adjusting your earlier guess by a constant (sometimes bigger than 1 sometimes less than 1).

Note: make sure to say "$\int_1^\infty 1/x^2 dx$ converges" not "$1/x^2$ converges"; it's important that you talk about the *integral* converging, not the function.

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$$\int_1^\infty \frac{1}{1+x} dx$$

Step 1: This looks like $\int_1^\infty 1/x$, so we expect this to diverge.

Step 2: We want to find a positive comparison function $f(x)$ so that

$$f(x) \leqslant 1/(x+1) \text{ for } x \geqslant 1$$

and $\int_1^\infty f(x)dx$ diverges. Our first guess is to try $1/x$, but this doesn't work. Instead, what we do is try $\frac{1}{2x}$. (Generally, this trick will work–if your comparison function doesn't work, try multiplying it by some constant, sometimes bigger than one, sometimes less than 1, and see if that will work.[1])

We check:

$$1/(2x) \leqslant 1/(x+1)$$

is the same as

$$(x+1) \leqslant 2x$$

by cross-multiplying, which is the same as

$$1 \leqslant x$$

Therefore, if $x \geqslant 1$, we have $(1/2x) \leqslant 1(x+1)$. Since $\int_1^\infty 1/(2x)dx$ diverges, and since $1/(2x) \leqslant 1/(x+1)$ for $x \geqslant 1$, we have that $\int_1^\infty 1/(x+1)dx$ diverges also.

[1]Note: It's **wrong** to say "Oh, this comparison function $1/x$" doesn't work, therefore it converges. Your initial guess in step 1 will almost always be correct (if you correctly figured out what the function is "like"). If your inequality is in the wrong direction for the comparison test, you have to find a new comparison function.

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$$\int_5^8 \frac{6}{\sqrt{t-5}}dt.$$

The problem is as $t \to 5$, since then the integrand doesn't make sense. Here it makes sense mostly just to calculate things directly.

$$\begin{aligned}\int_5^8 \frac{6}{\sqrt{t-5}}dt &= \lim_{b\to 5^+}\int_b^8 \frac{6}{\sqrt{t-5}}dt \\ &= \lim_{b\to 5^+}\int_{b-5}^3 \frac{6}{\sqrt{u}}du\end{aligned}$$

where we have made the substitution $u = t-5$.[2] Note that the limit as $b \to 5^+$ of $b-5$ is the same thing as the limit of a as $a \to 0^+$. So this is the same as

$$\lim_{a\to 0^+}\int_a^3 \frac{6}{\sqrt{u}}du = \int_0^3 \frac{6}{\sqrt{u}}du,$$

which converges, since this is like $\int_0^1 \frac{1}{x^p}$ for $p < 1$.

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$$\int_1^\infty \frac{1}{\sqrt{\theta^2+1}}d\theta.$$

Step one: as $\theta \to \infty$ the function is like $\frac{1}{\sqrt{\theta^2}} = 1/\theta$, so we expect the integral to diverge.

Step two: To use the comparison test, we need to find a positive comparison function $f(x)$ that is *smaller* than $\frac{1}{\sqrt{\theta^2+1}}$ and whose integral from 1 to ∞ diverges. Unfortunately, the obvious choice, $1/\theta$, will not be smaller than $\frac{1}{\sqrt{\theta^2+1}}$.

Instead, what we do is try $\frac{1}{2\theta}$, hoping that this will be smaller than $\frac{1}{\sqrt{\theta^2+1}}$, at least for large enough x.

We have

$$\frac{1}{2\theta} \leqslant \frac{1}{\sqrt{\theta^2+1}}$$

if and only if (by cross-multiplying)

$$\sqrt{\theta^2+1} \leqslant 2\theta,$$

which will be true if and only if

$$\theta^2 + 1 \leqslant 2\theta^2,$$

which will be true if and only if

$$1 \leqslant \theta^2.$$

[2]Note: whenever you have an improper integral where the problem is caused by something being zero in the denominator, like $\int_3^5 \frac{1}{x-3}dx$, there's often a substitution (e.g., $u = x - 3$) that will turn this into something like $\int_0^a \frac{1}{x^p}dx$.

So, as long as $\theta \geqslant 1$, we know that $\frac{1}{2\theta} \leqslant \frac{1}{\sqrt{\theta^2+1}}$.

Therefore,

$$\int_1^\infty \frac{1}{2\theta} d\theta \leqslant \int_1^\infty \frac{1}{\sqrt{\theta^2+1}} d\theta.$$

We know that $\int_1^\infty \frac{1}{2\theta} d\theta$ diverges, therefore $\int_1^\infty \frac{1}{\sqrt{\theta^2+1}} d\theta$ will diverge also.